FRW Universe Models in Conformally Flat Spacetime Coordinates
I: General Formalism

Øyvind Grøn* and Steinar Johannesen*

* Oslo University College, Department of Engineering, P.O.Box 4 St.Olavs Plass, N-0130 Oslo, Norway

Abstract The 3-space of a universe model is defined at a certain simultaneity. Hence space depends on which time is used. We find a general formula generating all known and also some new transformations to conformally flat spacetime coordinates. A general formula for the recession velocity is deduced.

1. Introduction.

Space-time and 3-space have very different characters according to the general theory of relativity. Spacetime is absolute, meaning that the properties of spacetime do not depend upon the reference frame. The 3-space, on the other hand is relative. Further on we will use the word *space* for the 3-space defined by a certain simultaneity.

A.J.S.Hamilton and J.P.Lisle [1] have recently presented a new way to conceptualize space in their river model of black holes. Similarly, in a cosmological context, we here define a flow of inertial reference particles as a continuum of free particles with a velocity field approaching asymptotically the large scale velocity field of the distant matter in the universe.

We see that there are two different ways of thinking about space: as a simultaneity space in 4-dimensional spacetime or as a flow of inertial frames. It may be useful to introduce separate names for these conceptions. We therefore make the following definitions. *Coordinate space* is a continuum of events taking place at a constant coordinate time. *Inertial flow* is a continuum of local inertial frames with vanishing velocity at a specified position. These frames consist of a flow of freely falling particles.

The spacetime outside a black hole provides an example of both. The coordinate space is defined by constant Schwarzschild time coordinate. The inertial flow is a continuum of inertial frames with vanishing velocity infinitely far from the black hole.

In cosmology one uses comoving coordinates in which freely moving reference particles have constant spatial coordinates. The coordinate time shown on locally Einstein synchronized clocks carried by the reference particles is called *cosmic time*, and the coordinate space defined by constant cosmic time is called the *cosmic space*. The cosmic inertial flow is usually called the *Hubble flow*.

Just as one can cut many different surfaces through a 3-dimensional body, one can define many different coordinate spaces in 4-dimensional spacetime. Flat spacetime, for example, can be separated in time and space in many different ways. Two of them are:

· The space in Minkowski spacetime. This is the coordinate space of a coordinate system comoving with free reference particles that constitute a non-expanding inertial flow.
· The private and public space of the Milne universe model. The private space of this universe model is just the Euclidean space in Minkowski spacetime, and the public space is the negatively curved coordinate space of a coordinate system comoving with a set of free particles constituting an expanding inertial flow, i.e. it is the cosmic space of an empty universe model.

Similarly, in the non-empty Friedmann-Robertson-Walker (FRW) universe models we shall define two types of coordinate space,
· cosmic space defined by constant cosmic time.
· conformal space defined by constant conformal time.

2. The line element of the FRW universe models

We shall here study FRW-universe models. The line element then takes the form

$$ds^2 = -dt^2 + a(t)^2[\,d\chi^2 + S_k(\chi)^2 d\Omega^2] \ . \tag{1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$. Here the dimensionless quantity χ is called the standard radial coordinate. We use a dimensionless cosmic time t. The scale factor $a(t)$ is normalized so that $a(t_0) = 1$ where t_0 is the present age of the universe. This means that $a(t)$ represents the distance between two objects at an arbitrary point of time relative to their distance at the present time. The parameter k characterizes the spatial curvature with $k = 1$ for a positively curved space, $k = 0$ for flat space, and $k = -1$ for a negatively curved space.

In the case of a universe model with curved space the length unit is equal to the present value r_0 of the spatial curvature radius,

$$r_0 = \frac{c}{H_0\sqrt{|1-\Omega_0|}} \ , \tag{2}$$

where the Hubble constant H_0 is the present value of the Hubble parameter, and $\Omega_0 \neq 1$ is the total energy density relative to the critical density at the point of time t_0. This expression for the spatial curvature follows from the Friedmann equations [2]. In a universe model with flat space, i.e. with $\Omega_0 = 1$, the unit of length is equal to the Hubble length c/H_0 of the universe. The unit of time is equal to the time taken by light to move a unit of length. This implies that (except in equations (2) and (5)) we use units so that $c = 1$. As a consequence of curvature isotropy in space the function S_k obeys [3,4]

$$S_k'^2 + kS_k^2 = 1 \ . \tag{3}$$

where ' denotes differentiation with respect to χ. Hence

$$S_k(\chi) = \begin{cases} \sin\chi & \text{for} \quad k=1 \quad , \quad 0<\chi<\pi \\ \chi & \text{for} \quad k=0 \quad , \quad 0<\chi<\infty \\ \sinh\chi & \text{for} \quad k=-1 \quad , \quad 0<\chi<\infty \end{cases} \ . \tag{4}$$

Furthermore $k=-1$ for $\Omega_0<1$, and $k=1$ for $\Omega_0>1$. Note that the unit of length may be written

$$l_0 = \begin{cases} \frac{c}{H_0}\sqrt{\frac{k}{\Omega_0-1}} & \text{for} \quad k \neq 0 \\ \frac{c}{H_0} & \text{for} \quad k = 0 \end{cases} \quad . \tag{5}$$

Each reference particle has constant radial coordinate χ. The reference particles may be thought of as clusters of galaxies. The distance between the reference particles increases due to the expansion of the universe. The general relativistic interpretation is that space expands [5]. For the line element (1) an elementary calculation shows that the Christoffel symbol Γ^{χ}_{tt} vanishes. Hence, particles with constant χ move freely. They define an inertial flow which is the Hubble flow of the universe model. The Hubble flow of these universe models expands if $a(t)$ is an increasing function of t.

The standard radial coordinate of the particle horizon is defined by

$$\chi_H = \int_{t_i}^{t} \frac{dt}{a(t)} \tag{6}$$

for universe models beginning at $t=t_i$, where different universe models may have $t_i=-\infty$, $t_i=0$ or $t_i>0$.

We now introduce a new time coordinate η, called the *parametric time* [4], defined by

$$\eta = \int_{t_1}^{t} \frac{dt}{a(t)} \tag{7}$$

where t_1 is an arbitrary point of cosmic time corresponding to $\eta=0$. The line element as expressed in terms of η takes the form

$$ds^2 = a(\eta)^2\,[\,-\,d\eta^2 + d\chi^2 + S_k(\chi)^2 d\Omega^2]\ . \tag{8}$$

The space defined by constant η is identical to the space defined by constant t. Note that $d\eta/dt<1$ for $t>t_0$ in expanding universe models. Hence the parametric time is shown on clocks that slow down with the expansion of the universe relative to the clocks showing cosmic time, and stops in the limit $a\to\infty$.

From the definitions (6) and (7) it follows that

$$\chi_H = \eta \tag{9}$$

for $t_i=-\infty$ and $t_i=0$. If the universe comes into existence at a point of time $t_i>0$, the horizon radius is given by

$$\chi_H = \int_{t_i}^{t} \frac{dt}{a(t)} = \eta - \eta(t_i) \tag{10}$$

The present velocity of a reference particle with $\chi=$ constant with respect to the observer at $\chi=0$ is given by Hubble's law

$$v_H = \widehat{H}_0 a(t_0)\chi = \widehat{H}_0\chi\ , \tag{11}$$

where

$$\widehat{H}_0 = l_0 H_0 \tag{12}$$

is a dimensionless Hubble parameter. Note that H_0 has the dimension inverse time. This means that with our dimensionless time coordinate

$$H_0 = \frac{1}{l_0 a(t_0)} \left(\frac{da}{dt}\right)_{t=t_0} \tag{13}$$

while

$$\widehat{H}_0 = \frac{1}{a(t_0)} \left(\frac{da}{dt}\right)_{t=t_0} \tag{14}$$

The velocity v_H of the Hubble flow is greater than 1 for $\chi > 1/\widehat{H}_0$.

From equations (5) and (12) it follows that $\widehat{H}_0 = 1$ for a flat universe. Then

$$\widehat{H} = \frac{H}{H_0} \tag{15}$$

i.e. the dimensionless Hubble parameter of a flat universe is equal to the ratio of the ordinary Hubble parameter at an arbitrary point of time and its present value. However, in a curved universe this normalization of the dimensionless Hubble parameter is not compatible with the corresponding normalization of the scale factor $a(t_0) = 1$.

Note also that since all distances are scaled by $a(t)$, the curvature radius of curved space at an arbitrary point of time is

$$r(t) = r_0 a(t) \tag{16}$$

Hence in a curved universe, the scale factor is equal to the dimensionless curvature radius $a(t) = r(t)/r_0$.

3. Conformal coordinates in FRW universe models

The FRW universe models have vanishing Weyl curvature tensor, and hence conformally flat spacetime. It is therefore possible to introduce coordinates (T, R) so that the line element has a conformally flat form [6-9]. These types of coordinates were introduced in the description of relativistic universe models by L.Infield and A.Schild [6], who sacrificed the advantage of the standard description where freely moving reference particles have constant coordinates, for a metric conformal to Minkowski spacetime, where the speed of light is constant. The coordinates (T, R) are called *conformally flat spacetime (CFS) coordinates* [10-12]. They have recently been applied by J.Garecki [13] to calculate the energy of matter dominated Friedmann universes. He has argued that in this connection there is a definite advantage in using CFS coordinates. Furthermore, G.U.Varieschi [14] has described universe models based on so called conformal gravity using CFS coordinates.

We shall here investigate the FRW universe models with reference to these coordinates. Then the line element has the form

$$ds^2 = A(T,R)^2(-dT^2 + dR^2 + R^2 d\Omega^2) = A(T,R)^2 ds_M^2 \ , \tag{17}$$

where $A(T,R)$ is the CFS scale factor, and ds_M^2 is the line element of the Minkowski spacetime. M.Ibison [15], K.Shankar and B.F.Whiting [16] and M.Iihoshi et al. [9] have shown that a general coordinate transformation that takes the line element (8) into the form (17) is

$$T = \frac{1}{2}\left[f(\eta+\chi)+g(\eta-\chi)\right] \quad , \quad R = \frac{1}{2}\left[f(\eta+\chi)-g(\eta-\chi)\right] , \tag{18}$$

where f and g are functions that must satisfy an identity deduced below. The transformation (18) can be described as a composition of three simple transformations. The first transforms from the coordinates η and χ in the line element (8) to light cone coordinates (null coordinates)

$$u = \eta + \chi \quad , \quad v = \eta - \chi \ . \tag{19}$$

This rotates the previous coordinate system by $-\pi/4$ and scales it by a factor $\sqrt{2}$. The scaling is performed for later convenience. The second transforms u and v to the coordinates

$$\tilde{u} = f(u) \quad , \quad \tilde{v} = g(v) \ . \tag{20}$$

Finally, we scale and rotate with the inverse of the transformation (19),

$$T = \frac{\tilde{u}+\tilde{v}}{2} \quad , \quad R = \frac{\tilde{u}-\tilde{v}}{2} \ . \tag{21}$$

Note that

$$T^2 - R^2 = \tilde{u}\tilde{v} \ . \tag{22}$$

Taking the differentials of T and R we get

$$-\,dT^2 + dR^2 = -d\tilde{u}d\tilde{v} = -f'(u)g'(v)\,du\,dv = f'(u)g'(v)(-d\eta^2 + d\chi^2) \ . \tag{23}$$

Comparing the expressions (8) and (17) for the line element and using the previous formula, we find

$$A(T,R)^2 = \frac{a(\eta)^2}{f'(u)g'(v)} \tag{24}$$

and

$$f'(u)g'(v)S_k(\chi)^2 = R^2 \ . \tag{25}$$

By (18) and (19) the last equation may be written as

$$f'(u)g'(v)\,S_k\left(\frac{u-v}{2}\right)^2 = \frac{1}{4}\left[f(u)-g(v)\right]^2 . \tag{26}$$

Substituting $v = u$ and utilizing that $S_k(0) = 0$, this equation gives $g(u) = f(u)$. Hence equation (26) reduces to

$$f'(u)f'(v)\,S_k\left(\frac{u-v}{2}\right)^2 = \frac{1}{4}\left[f(u)-f(v)\right]^2 , \tag{27}$$

and equation (18) takes the form

$$T = \frac{1}{2}\left[f(\eta+\chi)+f(\eta-\chi)\right] \quad , \quad R = \frac{1}{2}\left[f(\eta+\chi)-f(\eta-\chi)\right] . \tag{28}$$

Inserting $\chi = 0$ gives $T = f(\eta)$ and $R = 0$. Hence the physical interpretation of the function f is that it represents the transformation from parametric time to conformal time at $\chi = R = 0$. Different choices of the function f generate different types of conformal coordinates. The function f is assumed to be increasing, meaning that the conformal time proceeds in the same direction as the parametric time at $\chi = R = 0$, i.e. in the same direction as the cosmic time. We shall later introduce several generating functions f satisfying this relationship. Inserting equation (25) in (24) and demanding that $A(T, R) > 0$, we obtain the following expression for the CFS scale factor

$$A(T, R) = \frac{a(\eta(T,R))\, S_k(\chi(T,R))}{|R|} \; . \tag{29}$$

We shall now integrate equation (27), written in the form

$$\frac{1}{S_k\left(\frac{u-v}{2}\right)^2} = \frac{4f'(u)f'(v)}{[f(u)-f(v)]^2} \; . \tag{30}$$

For this purpose we introduce the function

$$I_k(x) = \begin{cases} \cot x & \text{for} \quad k = 1 \\ 1/x & \text{for} \quad k = 0 \\ \coth x & \text{for} \quad k = -1 \end{cases} \tag{31}$$

so that $I_k'(x) = -S_k(x)^{-2}$. Inserting $v = a$ where a is an arbitrary constant in equation (30) and integrating leads to

$$f(x) - f(a) = \frac{2f'(a)}{b + I_k\left(\frac{x-a}{2}\right)} \; , \tag{32}$$

where b is a constant of integration. This generalises a corresponding expression deduced by Iihoshi et al. [9] who have put $a = 0$. As will be shown below, our more general choice gives us a new solution not obtainable with $a = 0$.

Equation (30) has some interesting properties. It is invariant with respect to an additive and a multiplicative constant on the function f. Hence $f(a)$ and $f'(a)$ may be chosen as arbitrary constants. This gives the solution

$$f(x) = c\left[b + I_k\left(\frac{x-a}{2}\right)\right]^{-1} + d \tag{33}$$

for $x \neq a$, where c and d are arbitrary constants. Demanding that the function f is continuously differentiable, we must define $f(a) = d$. In appendix A we demonstrate that this function satisfies the equation (27). The expression (33) contains generating functions of many transformations to conformally flat coordinates considered previously, as well as new ones.

We shall now show that the coordinate differentials transform as a combination of a Lorentz transformation and a scaling. Consider a comoving particle P_H in the Hubble flow, keeping χ constant. We shall relate this particle to one at rest in the conformal coordinate system, $R =$ constant, by a local Lorentz transformation. The partial derivatives of the coordinates T and R are

$$\frac{\partial T}{\partial \eta} = \frac{\partial R}{\partial \chi} = \frac{1}{2}\left[f'(u) + f'(v)\right] \tag{34}$$

and

$$\frac{\partial R}{\partial \eta} = \frac{\partial T}{\partial \chi} = \frac{1}{2}\left[f'(u) - f'(v)\right] . \tag{35}$$

The *recession velocity* in cosmology is the coordinate velocity of particles with $\chi =$ constant, i.e. it is the coordinate velocity of the Hubble flow. Obviously, the recession velocity of the Hubble flow vanishes in the cosmic coordinate system. G.Endean has introduced the recession velocity in the CFS coordinate system [10]. It is given by

$$V = \tanh\theta = \left(\frac{dR}{dT}\right)_{\chi=\text{constant}} = \frac{f'(u)-f'(v)}{f'(u)+f'(v)} . \tag{36}$$

where θ is the rapidity of the particle P_H in the CFS system. It may be noted from equation (36) that this velocity cannot exceed that of light. From this equation it also follows that the Lorentz factor is

$$\gamma = \cosh\theta = \frac{f'(u)+f'(v)}{2\left[f'(u)f'(v)\right]^{1/2}} , \tag{37}$$

and

$$\gamma V = \sinh\theta = \frac{f'(u)-f'(v)}{2\left[f'(u)f'(v)\right]^{1/2}} . \tag{38}$$

Hence

$$\left[f'(u)f'(v)\right]^{-1/2} dT = \cosh\theta\, d\eta + \sinh\theta\, d\chi \tag{39}$$

and

$$\left[f'(u)f'(v)\right]^{-1/2} dR = \sinh\theta\, d\eta + \cosh\theta\, d\chi . \tag{40}$$

The differentials of the coordinates used by the two observers are therefore related by

$$\begin{bmatrix} dT \\ dR \end{bmatrix} = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} d\eta \\ d\chi \end{bmatrix} , \tag{41}$$

which corresponds to a composition of a Lorentz transformation with rapidity θ and a scaling with the factor

$$B = \left[f'(u)f'(v)\right]^{1/2} = \frac{a}{A} , \tag{42}$$

which we call the *relative scale factor*. It can also be expressed in terms of T and R by

$$B = \frac{|R|}{S_k(\chi(T,R))} . \tag{43}$$

Note that the scale factor $A(T,R)$ in the line element for the CFS system depends upon the radial coordinate R. Hence as described in the CFS coordinates the universe looks inhomogeneous. This has the following explanation. Due to the relativity of simultaneity and the relative motion of the reference particles, $R =$ constant, of the CFS system and those of the cosmic system, $\chi =$ constant, the CFS space represents a different simultaneity space than the cosmic space. The universe is homogeneous, but time dependent in the cosmic system. Thus the time dependence of the cosmic system is transformed to a time and space dependence in the CFS system.

By means of the expression (33) for the generating function, one may deduce the following general formula for the recession velocity in conformally flat space (see Appendix C),

$$V = \left(\frac{dR}{dT}\right)_{\chi=\text{constant}} = \frac{f'(f^{-1}(T+R)) - f'(f^{-1}(T-R))}{f'(f^{-1}(T+R)) + f'(f^{-1}(T-R))} \tag{44}$$

where

$$f'(f^{-1}(x)) = \frac{1}{2c}[\{b(x-d)-c\}^2 + k(x-d)^2] \ . \tag{45}$$

Inserting the expression (45) into equation (44), we obtain

$$V = \frac{2R[(b^2+k)(T-d) - bc]}{(b^2+k)[(T-d)^2+R^2] - c[2b(T-d)-c]} \ . \tag{46}$$

From equation (14) and the relation $dt = ad\eta$ it follows that

$$\widehat{H} = \frac{1}{a^2}\frac{da}{d\eta} \ . \tag{47}$$

It will later be shown that the parametric time η may be used as a CFS time coordinate for flat universe models. This motivates the following definition of the Hubble parameter in an arbitrary CFS coordinate system

$$H_R = \frac{1}{A^2}\frac{\partial A}{\partial T} \ . \tag{48}$$

This definition means that H_R represents the expansion or the contraction of the CFS flow defined by reference particles with $R =$ constant. It should be noted, however, that these particles are not freely falling. They do not constitute an inertial flow.

From equation (36) we find that the Doppler shift factor due to the recession velocity is

$$D = \sqrt{\frac{1+V}{1-V}} = \sqrt{\frac{f'(u)}{f'(v)}} \ . \tag{49}$$

Using the expression (33) for the generating function we arrive at the following general expression of the Doppler factor

$$D(T,R) = \left\{\frac{k(T+R-d)^2 + [b(T+R-d)-c]^2}{k(T-R-d)^2 + [b(T-R-d)-c]^2}\right\}^{1/2} \ . \tag{50}$$

The total redshift z of an object at R emitting light at a time T as observed at $R = 0$ at a time T_0 is given by [6]

$$1 + z = D(T,R)\,\frac{A(T_0,0)}{A(T,R)} \tag{51}$$

where $R = T_0 - T$, since light moves with constant velocity in the CFS system.

4. Conclusion

The main result of the present paper is the general expression (33) generating all transformations of the form (18) from FRW standard coordinates to CFS coordinates. The

physical interpretation of these transformations is given in (41), showing that is it a composition of a Lorentz transformation and a scaling. An advantage of the conformally flat form of the line element is that the coordinate velocity of light is isotropic and has the constant value c everywhere.

Furthermore, in appendix B we show how one can construct conformal transformations between FRW spacetimes of arbitrary spatial curvature. Specifically we may want to transform an arbitrary form of the line element of a FRW universe to a form with a conformal factor times the line element of a static FRW universe with scale factor equal to 1 and a specified curvature. One may always start the construction by introducing a parametric time. The result is a line element equal to a conformal factor times a static FRW universe with the same spatial curvature. The curvature can then be changed using appendix B in the following way. First one transforms to a conformally flat form. Then one transforms to the static form with the chosen spatial curvature using the inverse of one of the transformations obtained by the formula (33). An important application is in the construction of Penrose diagrams utilizing a form of the line element with a conformal factor times the line element of Einstein's static universe having positive spatial curvature.

Further applications of the formalism developed in the present paper will be given in later articles in this series.

Appendix A. Proof that f given in equation (33) satisfies equation (27)

We first introduce the function

$$C_k(x) = I_k(x)\, S_k(x) = \begin{cases} \cos x & \text{for} \quad k=1 \\ 1 & \text{for} \quad k=0 \\ \cosh x & \text{for} \quad k=-1 \end{cases} \tag{52}$$

so that $S_k'(x) = C_k(x)$. This function also satisfies the relation

$$S_k(u-v) = S_k(u)\, C_k(v) - C_k(u)\, S_k(v) \ . \tag{53}$$

Furthermore

$$f'(x) = \frac{c}{2}\left[b\, S_k\left(\frac{x-a}{2}\right) + C_k\left(\frac{x-a}{2}\right)\right]^{-2} \ . \tag{54}$$

Inserting the function f in equation (27) leads to

$$\begin{aligned} \frac{[f(u)-f(v)]^2}{4f'(u)f'(v)} &= \left\{\left[b + I_k\left(\frac{u-a}{2}\right)\right]^{-1} - \left[b + I_k\left(\frac{v-a}{2}\right)\right]^{-1}\right\}^2 \\ &\quad \left[b + I_k\left(\frac{u-a}{2}\right)\right]^2 \left[b + I_k\left(\frac{v-a}{2}\right)\right]^2 S_k\left(\frac{u-a}{2}\right)^2 S_k\left(\frac{v-a}{2}\right)^2 \\ &= \left[I_k\left(\frac{v-a}{2}\right) - I_k\left(\frac{u-a}{2}\right)\right]^2 S_k\left(\frac{u-a}{2}\right)^2 S_k\left(\frac{v-a}{2}\right)^2 \ . \end{aligned} \tag{55}$$

Using the relation (53) we obtain

$$\frac{[f(u)-f(v)]^2}{4f'(u)f'(v)} = \left[S_k\left(\frac{u-a}{2}\right) C_k\left(\frac{v-a}{2}\right) - C_k\left(\frac{u-a}{2}\right) S_k\left(\frac{v-a}{2}\right)\right]^2 = S_k\left(\frac{u-v}{2}\right)^2 \tag{56}$$

which was to be shown.

Appendix B. Composition of generating functions

In section 3 we have seen that it is possible to transform away spatial curvature. In the same way we obtain a more general transformation from spaces with curvature k_1 to spaces with curvature k_2 if the generating function f satisfies the relation

$$f'(u)f'(v)\, S_{k_1}\left(\tfrac{u-v}{2}\right)^2 = S_{k_2}\left(\tfrac{f(u)-f(v)}{2}\right)^2 \ . \tag{57}$$

We shall now prove the following rule of composition for generating functions. Let f be a generating function between spaces with curvatures k_1 and k_2, and g a generating function between spaces with curvatures k_2 and k_3, both satisfying equation (57). Then the composition h of g and f, $h(x) = g(f(x))$, is a generating function between spaces with curvatures k_1 and k_3. The rule is proved by utilizing the chain rule of differentiation.

$$\begin{aligned} &S_{k_3}\left(\tfrac{h(u)-h(v)}{2}\right)^2 = S_{k_3}\left(\tfrac{g(f(u))-g(f(v))}{2}\right)^2 = g'(f(u))\, g'(f(v))\, S_{k_2}\left(\tfrac{f(u)-f(v)}{2}\right)^2 \\ &= g'(f(u))\, g'(f(v))\, f'(u)\, f'(v)\, S_{k_1}\left(\tfrac{u-v}{2}\right)^2 = h'(u)\, h'(v)\, S_{k_1}\left(\tfrac{u-v}{2}\right)^2 \ . \end{aligned} \tag{58}$$

We also have the following rule for the inverse of a generating function. Let f be an invertible generating function between spaces with curvatures k_1 and k_2 satisfying equation (57). Then the inverse funtion $g = f^{-1}$ is a generating function between spaces with curvatures k_2 and k_1. For if $U = f(u)$ and $V = f(v)$, then

$$\begin{aligned} &S_{k_1}\left(\tfrac{g(U)-g(V)}{2}\right)^2 = S_{k_1}\left(\tfrac{u-v}{2}\right)^2 = \frac{1}{f'(u)}\ \frac{1}{f'(v)}\ S_{k_2}\left(\tfrac{f(u)-f(v)}{2}\right)^2 \\ &= g'(U)\ g'(V)\ S_{k_2}\left(\tfrac{U-V}{2}\right)^2 \ . \end{aligned} \tag{59}$$

Appendix C. A general formula for the recession velocity

The recession velocity is given in equation (36) which may be written

$$V = \left(\frac{dR}{dT}\right)_{\chi=\text{constant}} = \frac{f'(f^{-1}(T+R)) - f'(f^{-1}(T-R))}{f'(f^{-1}(T+R)) + f'(f^{-1}(T-R))} \tag{60}$$

Hence we need to calculate the function $y = f'(f^{-1}(x))$ using the expression (33). This gives

$$f^{-1}(x) = 2\, I_k^{-1}\left(\tfrac{c}{x-d} - b\right) + a \tag{61}$$

where

$$I_k^{-1}(x) = \begin{cases} \operatorname{arccot} x & \text{for} \quad k = 1 \\ 1/x & \text{for} \quad k = 0 \\ \operatorname{arccoth} x & \text{for} \quad k = -1 \end{cases} \tag{62}$$

In order to combine this with formula (54) for f', we need the following formulae

$$S_k(I_k^{-1}(x)) = \begin{cases} \frac{1}{\sqrt{x^2+1}} & \text{for} \quad k = 1 \\ 1/x & \text{for} \quad k = 0 \\ \frac{1}{\sqrt{x^2-1}} & \text{for} \quad k = -1 \end{cases} \tag{63}$$

and

$$C_k(I_k^{-1}(x)) = x\, S_k(I_k^{-1}(x)) \tag{64}$$

In the arguments of S_k and C_k of the formula (54) we use that

$$\tfrac{f^{-1}(x)-a}{2} = I_k^{-1}\left(\tfrac{c}{x-d} - b\right) \ . \tag{65}$$

Combining this with equation (65) we get

$$f'(f^{-1}(x)) = \tfrac{c}{2}\left(\tfrac{c}{x-d}\right)^{-2} S_k\left(I_k^{-1}\left(\tfrac{c}{x-d} - b\right)\right)^{-2} \ . \tag{66}$$

By means of equation (63) we finally obtain

$$f'(f^{-1}(x)) = \frac{1}{2c}[\{b(x-d) - c\}^2 + k(x-d)^2] \tag{67}$$

References

1. A.J.S.Hamilton and J.P.Lisle, *The river model of black holes*, Am.J.Phys. **76**, 519 - 532 (2008).

2. Ø.Grøn and S.Hervik, Einstein's General Theory of Relativity, Springer, (2007), ch.11.

3. Ø.Grøn, Lecture Notes on the General Theory of Relativity, Springer, 2009, p 208.

4. C.F.Sopuerta, *Stationary generalized Kerr-Schild spacetimes*, J.Math.Phys. **39**, 1024 - 1039 (1998).

5. Ø.Grøn and Ø.Elgarøy, *Is space expanding in the Friedman universe models?*, Am.J. Phys. **75**, 151 - 157 (2007). Further references on this topic are found here.

6. L.Infield and A.Schild, *A New Approach to Kinematic Cosmology*, Phys.Rev. **68**, 250 - 272 (1945).

7. G.E.Tauber, *Expanding Universe in Conformally Flat Coordinates*, J.Math.Phys. **8**, 118 - 123 (1967).

8. V.F.Mukhanov, *Physical foundations of cosmology*, Cambridge University Press, (2005).

9. M.Iihoshi, S.V.Ketov and A.Morishita, *Conformally Flat FRW Metrics*, Prog.Theor. Phys. **118**, 475 - 489 (2007)

10. G.Endean, *Redshift and the Hubble Constant in Conformally Flat Spacetime*, The Astrophysical Journal **434**, 397 - 401 (1994).

11. G.Endean, *Resolution of Cosmological age and redshift-distance difficulties*, Mon.Not. R.Astron.Soc. **277**, 627 - 629 (1995).

12. G.Endean, *Cosmology in Conformally Flat Spacetime*, The Astrophysical Journal **479**, 40 - 45 (1997).

13. J.Garecki, *On Energy of the Friedmann Universes in Conformally Flat Coordinates*, arXiv:0708.2783.

14. G.U.Varieschi, *A Kinematical Approach to Conformal Cosmology*, Gen.Rel.Grav. **42**, 929 - 974 (2010).

15. M.Ibison, *On the conformal forms of the Robertson-Walker metric*, J.Math.Phys. **48**, 122501-1 – 122501-23 (2007).

16. K.Shankar and B.F.Whiting, *Conformal coordinates for a constant density star*, arXiv:0706.4324.